Exercise 9.7.1

For a homogeneous spherical solid with constant thermal diffusivity, K, and no heat sources, the equation of heat conduction becomes

$$\frac{\partial T(r,t)}{\partial t} = K\nabla^2 T(r,t).$$

Assume a solution of the form

$$T = R(r)T(t)$$

and separate variables. Show that the radial equation may take on the standard form

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + \alpha^2 r^2 R = 0,$$

and that $\sin\alpha r/r$ and $\cos\alpha r/r$ are its solutions.

[TYPO: T represents the temperature. Use a different variable Θ for the separated function of t.]

Solution

Because the solid is spherical, expand the Laplacian operator in spherical polar coordinates (r, θ, φ), where θ is the angle from the polar axis.

$$\frac{\partial T(r,t)}{\partial t} = K\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T(r,t)}{\partial r}\right) + \underbrace{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T(r,t)}{\partial\theta}\right)}_{=0} + \underbrace{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T(r,t)}{\partial\varphi^2}}_{=0}\right]$$

T is only a function of r and t, so the angular derivatives vanish.

$$\frac{\partial T}{\partial t} = \frac{K}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)$$

The equation of heat conduction is linear and homogeneous, so the method of separation of variables can be applied to solve it. Assume a product solution of the form $T(r,t) = R(r)\Theta(t)$ and substitute it into the PDE.

$$\frac{\partial}{\partial t}[R(r)\Theta(t)] = \frac{K}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}[R(r)\Theta(t)]\right]$$

$$R\frac{d\Theta}{dt} = \Theta\frac{K}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)$$

Divide both sides by $KR(r)\Theta(t)$. (The final answer for $T(r,t)$ will be the same regardless which side K is on.)

$$\underbrace{\frac{1}{K\Theta}\frac{d\Theta}{dt}}_{\text{function of } t} = \underbrace{\frac{1}{r^2 R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{\text{function of } r}$$

The only way a function of t can be equal to a function of r is if both are equal to a constant λ.

$$\frac{1}{K\Theta}\frac{d\Theta}{dt} = \frac{1}{r^2 R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \lambda$$

As a result of applying the method of separation of variables, the equation of conduction has reduced to two ODEs—one in r and one in t.

$$\left.\begin{aligned} \frac{1}{K\Theta}\frac{d\Theta}{dt} &= \lambda \\ \frac{1}{r^2R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) &= \lambda \end{aligned}\right\}$$

Solve the first ODE for Θ.

$$\frac{d\Theta}{dt} = K\lambda\Theta$$

The general solution is written in terms of the exponential function.

$$\Theta(t) = C_1 e^{K\lambda t}$$

In order for $T(r,t)$ to remain bounded as $t \to \infty$, we require that λ be either zero or negative. Suppose first that λ is zero: $\lambda = 0$. The ODE for R becomes

$$\frac{1}{r^2R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = 0.$$

Multiply both sides by r^2R.

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = 0.$$

Integrate both sides with respect to r.

$$r^2\frac{dR}{dr} = C_2$$

Divide both sides by r^2.

$$\frac{dR}{dr} = \frac{C_2}{r^2}$$

Integrate both sides with respect to r once more.

$$R(r) = -\frac{C_2}{r} + C_3$$

Note that this is the steady-state temperature profile in a spherical geometry. With two boundary conditions, one could determine the constants, C_2 and C_3. Suppose secondly that λ is negative: $\lambda = -\alpha^2$. The ODE for R becomes

$$\frac{1}{r^2R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\alpha^2.$$

Multiply both sides by r^2R.

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\alpha^2r^2R$$

Use the product rule to expand the left side.

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} = -\alpha^2r^2R$$

The radial equation is thus

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + \alpha^2r^2R = 0.$$

Make the change of variables,

$$R = \frac{W}{r}.$$

Find the derivatives of R in terms of this new variable.

$$\frac{dR}{dr} = -\frac{W}{r^2} + \frac{W'}{r}$$

$$\frac{d^2R}{dr^2} = \frac{2}{r^3}W - \frac{W'}{r^2} - \frac{W'}{r^2} + \frac{W''}{r} = \frac{2}{r^3}W - \frac{2}{r^2}W' + \frac{1}{r}W''$$

Substitute these formulas into the radial equation to obtain an ODE for W.

$$r^2\left(\frac{2}{r^3}W - \frac{2}{r^2}W' + \frac{1}{r}W''\right) + 2r\left(-\frac{W}{r^2} + \frac{W'}{r}\right) + \alpha^2 r^2\left(\frac{W}{r}\right) = 0$$

$$\frac{2}{r}W - 2W' + rW'' - \frac{2}{r}W + 2W' + \alpha^2 rW = 0$$

$$rW'' + \alpha^2 rW = 0$$

Divide both sides by r.

$$W'' + \alpha^2 W = 0$$

The general solution is written in terms of sine and cosine.

$$W(r) = C_4 \cos\alpha r + C_5 \sin\alpha r$$

Therefore, since $R = W/r$,

$$R(r) = C_4\frac{\cos\alpha r}{r} + C_5\frac{\sin\alpha r}{r}.$$